Solution Sheet 1

Exercise 1.1.

For $a, b \in \mathbb{R}$, let $\alpha \in L^1([a,b];\mathbb{R})$, $\beta \in C([a,b];[0,\infty))$ and $u \in C([a,b];\mathbb{R})$ satisfy

$$u_t \leq \alpha_t + \int_a^t \beta_s u_s ds \tag{1}$$

for all $t \in [a,b]$. Prove the following versions of Grönwall's Inequality (Lemmas 1.2.3, 1.2.4):

1. For all $t \in [a,b]$,
$$u_t \leq \alpha_t + \int_a^t \alpha_s \beta_s e^{\int_s^t \beta_r dr} ds.$$

2. Under the additional assumption that α is constant,[1] then for all $t \in [a,b]$,
$$u_t \leq \alpha e^{\int_a^t \beta_r dr}.$$

Proof. We begin with the first part. For $s \in [a,b]$ define

$$x_s = e^{-\int_a^s \beta_r dr} \int_a^s \beta_r u_r dr$$

thus

$$\begin{aligned}
\dot{x}_s &= \left(\frac{d}{ds} e^{-\int_a^s \beta_r dr}\right) \int_a^s \beta_r u_r dr + e^{-\int_a^s \beta_r dr} \left(\frac{d}{ds} \int_a^s \beta_r u_r dr\right) \\
&= -\beta_s e^{-\int_a^s \beta_r dr} \int_a^s \beta_r u_r dr + e^{-\int_a^s \beta_r dr} \beta_s u_s \\
&= \left(u_s - \int_a^s \beta_r u_r dr\right) \beta_s e^{-\int_a^s \beta_r dr} \\
&\leq \alpha_s \beta_s e^{-\int_a^s \beta_r dr}
\end{aligned}$$

where we have used the assumption (1) in the last line. Observing that $x_a = 0$, then

$$x_t = x_t - x_a = \int_a^t \dot{x}_s ds \leq \int_a^t \alpha_s \beta_s e^{-\int_a^s \beta_r dr} ds. \tag{2}$$

Rearranging from the definition of x and plugging in (2), we see that

$$\int_a^t \beta_s u_s ds = e^{\int_a^t \beta_r dr} x_t \leq e^{\int_a^t \beta_r dr} \int_a^t \alpha_s \beta_s e^{-\int_a^s \beta_r dr} ds = \int_a^t \alpha_s \beta_s e^{\int_s^t \beta_r dr} ds.$$

Substituting this result into (1) completes the proof of the first part. For the second, we use that α is constant and the first part to see that

$$u_t \leq \alpha + \alpha \int_a^t \beta_s e^{\int_s^t \beta_r dr} ds. \tag{3}$$

[1] In fact one only needs that α is non-decreasing, and the result holds for α_t replacing α.

We observe that, on $[a, t]$,

$$\beta_s e^{\int_s^t \beta_r dr} = \frac{d}{ds}\left(-e^{\int_s^t \beta_r dr}\right)$$

hence

$$\int_a^t \beta_s e^{\int_s^t \beta_r dr} ds = -1 + e^{\int_a^t \beta_r dr}$$

which gives the result upon substitution into (3). In the case where α is non-decreasing simply repace α by α_t in (3) and the rest follows identically.

□

Exercise 1.2.

Give an example of a stochastic process X whose law is constant in time, but is not stationary (Definition 2.2.5).

Proof. Our example is $X_t = \frac{1}{\sqrt{t}} W_t$ where W is a standard Brownian Motion. It is clear that for all $t > 0$, $X_t \sim N(0, 1)$, though we show that the process is non-stationary by considering the covariance between X_s, X_t with $s < t$:

$$\operatorname{Cov}(X_s, X_t) = \mathbb{E}(X_s X_t) = \frac{1}{\sqrt{st}} \mathbb{E}(W_s W_t) = \frac{1}{\sqrt{st}} s.$$

It is clear that this does not depend on only $t - s$, and we can consider the cases $s = 1$, $t = 2$ and $s = 2$, $t = 3$ as a counterexample. □

Exercise 1.3.

Consider an ODE

$$\dot{x}_t = f(x_t)$$

in $\mathbb{R}^d$ with Euclidean norm $\|\cdot\|$, $d \geq 1$, where f is Lipschitz with Lipschitz constant K in $\mathbb{R}^d$. We introduce the flow map associated to the ODE, which is a function Φ defined for times s, t and a point $u \in \mathbb{R}^d$ as the solution x of the ODE at time t with 'initial condition' $x_s = u$. We denote this by $\Phi_{s,t}(u)$ and in the case where $s = 0$, simply $\Phi_t(u)$. Prove the following:

1. For any $s < t \in \mathbb{R}$, $\Phi_{s,t}(u) = \Phi_{t-s}(u)$;
2. For any $0 \leq s < t$, $\Phi_{s,t}(\Phi_s(u)) = \Phi_t(u)$;
3. For any $t \geq 0$,

$$\sup_{r \in [0,t]} \|\Phi_r(u) - \Phi_r(v)\| \leq \|u - v\| e^{Kt}.$$

Proof. We prove the parts in turn:

1. We use the explicit forms of solutions, namely that

$$\Phi_{s,t}(u) = u + \int_s^t f(\Phi_{s,r}(u))\, dr \qquad \text{and} \qquad \Phi_{t-s}(u) = u + \int_0^{t-s} f(\Phi_r(u))\, dr.$$

To compare the integrals we use a change of variables, writing

$$\int_0^{t-s} f(\Phi_r(u))\, dr = \int_s^t f(\Phi_{r-s}(u))\, dr$$

so that

$$\Phi_{s,t}(u) - \Phi_{t-s}(u) = \int_s^t f\left(\Phi_{s,r}(u)\right) - f\left(\Phi_{r-s}(u)\right) dr.$$

Taking the norm and using the Lipschitz property of f,

$$\begin{aligned}\|\Phi_{s,t}(u) - \Phi_{t-s}(u)\| &\leq \int_s^t \|f\left(\Phi_{s,r}(u)\right) - f\left(\Phi_{r-s}(u)\right)\| dr \\ &\leq K \int_s^t \|\Phi_{s,r}(u) - \Phi_{r-s}(u)\| dr.\end{aligned}$$

By setting $\psi_t = \|\Phi_{s,t}(u) - \Phi_{t-s}(u)\|$, the above reads

$$\psi_t \leq K \int_s^t \psi_r dr$$

from which we apply Grönwall's Inequality where $\alpha = 0$ to conclude that $\psi_t = 0$ hence the result.

2. We observe that

$$\Phi_{s,t}\left(\Phi_s(u)\right) = \Phi_s(u) + \int_s^t f\left(\Phi_{s,r}\left(\Phi_s(u)\right)\right) dr = u + \int_0^s f\left(\Phi_r(u)\right) dr + \int_s^t f\left[\Phi_{s,r}\left(\Phi_s(u)\right)\right] dr$$

and

$$\Phi_t(u) = u + \int_0^t f\left(\Phi_r(u)\right) dr = u + \int_0^s f\left(\Phi_r(u)\right) dr + \int_s^t f\left(\Phi_r(u)\right) dr.$$

Taking the difference of these expressions,

$$\Phi_{s,t}\left(\Phi_s(u)\right) - \Phi_t(u) = \int_s^t f\left[\Phi_{s,r}\left(\Phi_s(u)\right)\right] - f\left(\Phi_r(u)\right) dr$$

from which point the proof follows exactly as the previous part, taking the norm, using the Lipschitz property and applying Grönwall.

3. We have that

$$\Phi_r(u) = u + \int_0^r f(\Phi_s(u)) ds \qquad \text{and} \qquad \Phi_r(v) = v + \int_0^r f(\Phi_s(v)) ds.$$

Therefore,

$$\Phi_r(u) - \Phi_r(v) = u - v + \int_0^r f(\Phi_s(u)) - f(\Phi_s(v)) ds.$$

By taking norms on both sides, using the triangle inequality and bounding norm of integral by integral of norm,

$$\begin{aligned}\|\Phi_r(u) - \Phi_r(v)\| &\leq \|u - v\| + \int_0^r \|f(\Phi_s(u)) - f(\Phi_s(v))\| ds \\ &\leq \|u - v\| + \int_0^r K\|\Phi_s(u) - \Phi_s(v)\| ds\end{aligned}$$

having also used the Lipschitz assumption in the final line. Now we take the supremum over $r \in [0,t]$ on both sides, and further bound the integrand by its supremum to see that

$$\begin{aligned}\sup_{r\in[0,t]} \|\Phi_r(u) - \Phi_r(v)\| &\leq \|u-v\| + \int_0^t K\|\Phi_s(u) - \Phi_s(v)\| ds \\ &\leq \|u-v\| + \int_0^t K \sup_{r\in[0,s]} \|\Phi_r(u) - \Phi_r(v)\| ds.\end{aligned}$$

This is of the form (1) for $u_t = \sup_{r\in[0,t]} \|\Phi_r(u) - \Phi_r(v)\|$, $\alpha_t = \alpha = \|u-v\|$ and $\beta_t = K$. By applying the second statement of Exercise 1.1, then

$$\sup_{r\in[0,t]} \|\Phi_r(u) - \Phi_r(v)\| \leq \|u-v\| e^{\int_0^t K ds} = \|u-v\| e^{Kt}$$

as required.

□

Exercise 1.4.

Suppose we have a pair of real valued ODEs,

$$\begin{aligned}\dot{x}_t &= f(x_t, z^1) \\ \dot{y}_t &= f(y_t, z^2)\end{aligned}$$

where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is K-Lipschitz in each variable, uniformly across the other variable: that is for every $a \in \mathbb{R}$ the functions $f(a,\cdot)$ and $f(\cdot,a)$ are both K-Lipschitz. Here $z^1, z^2 \in \mathbb{R}$ are given. Let Φ^x denote the flow for x and Φ^y denote the flow for y. Prove that for any $t \geq 0$,

$$\sup_{r\in[0,t]} |\Phi^x_r(u^1) - \Phi^y_r(u^2)| \leq \left(|u^1 - u^2| + K|z^1 - z^2|t\right) e^{Kt}.$$

Proof. The difference equation for $r \geq 0$ is

$$\Phi^x_r(u^1) - \Phi^y_r(u^2) = u^1 - u^2 + \int_0^r f\left(\Phi^x_s(u), z^1\right) - f\left(\Phi^y_s(v), z^2\right) ds.$$

To use the Lipschitz property of f we rewrite

$$f\left(\Phi^x_s(u^1), z^1\right) - f\left(\Phi^y_s(u^2), z^2\right) = f\left(\Phi^x_s(u^1), z^1\right) - f\left(\Phi^x_s(u^1), z^2\right) + f\left(\Phi^x_s(u^1), z^2\right) - f\left(\Phi^y_s(u^2), z^2\right)$$

after which we take the absolute value of the difference equation and simplify to

$$|\Phi^x_r(u^1) - \Phi^y_r(u^2)| \leq |u^1 - u^2| + \int_0^r |f\left(\Phi^x_s(u^1), z^1\right) - f\left(\Phi^x_s(u^1), z^2\right)| + |f\left(\Phi^x_s(u^1), z^2\right) - f\left(\Phi^y_s(u^2), z^2\right)| ds.$$

Using the uniform Lipschitz property,

$$\begin{aligned}|\Phi^x_r(u^1) - \Phi^y_r(u^2)| &\leq |u^1 - u^2| + K\int_0^r |z^1 - z^2| + |\Phi^x_s(u^1) - \Phi^y_s(u^2)| ds \\ &\leq |u^1 - u^2| + K|z^1 - z^2|r + K\int_0^r |\Phi^x_s(u^1) - \Phi^y_s(u^2)| ds.\end{aligned}$$

We now take the supremum over $r \in [0,t]$ and again take a supremum inside the integral as in the previous exercise, obtaining

$$\sup_{r\in[0,t]} |\Phi^x_r(u^1) - \Phi^y_r(u^2)| \leq |u^1 - u^2| + K|z^1 - z^2|t + K\int_0^t \sup_{r\in[0,s]} |\Phi^x_r(u^1) - \Phi^y_r(u^2)|ds.$$

We apply the Grönwall Inequality of Exercise 1.1, assertion 2, for the function

$$\alpha_t = |u^1 - u^2| + K|z^1 - z^2|t$$

which is non-decreasing. Thus,

$$\sup_{r\in[0,t]} |\Phi^x_r(u^1) - \Phi^y_r(u^2)| \leq \left(|u^1 - u^2| + K|z^1 - z^2|t\right) e^{Kt}$$

as required.

□

Exercise 1.5.

Consider the real valued ODE

$$\dot{x}_t = -\alpha x_t + \beta \dot{f}_t$$

where $f \in C^1([0,\infty);\mathbb{R})$, $\alpha \in \mathbb{R}$ and initial condition $x_0 \in \mathbb{R}$. Show that for any $t \geq 0$ we have the identity

$$x_t = e^{-\alpha t}x_0 + \beta\int_0^t e^{-\alpha(t-s)}\dot{f}_s ds. \tag{4}$$

In particular,

$$x_t = e^{-\alpha t}x_0 + \beta\int_0^t e^{-\alpha(t-s)}df_s$$

where the last term is defined in the Riemann-Stieltjes sense.

Proof. We employ the integrating factor method, rearranging the equation and multiplying by $e^{\int_0^t \alpha ds} = e^{\alpha t}$ to obtain

$$e^{\alpha t}\dot{x}_t + e^{\alpha t}\alpha x_t = e^{\alpha t}\beta\dot{f}_t.$$

By design of this method the left hand side is simply $\frac{d}{dt}\left(e^{\alpha t}x_t\right)$ so integrating both sides over $[0,t]$ we obtain

$$e^{\alpha t}x_t - x_0 = \beta\int_0^t e^{\alpha r}\dot{f}_r dr.$$

Dividing through by $e^{-\alpha t}$ then gives (4). The second identity follows by standard properties of the Riemann-Stieltjes integral (f is C^1 so certainly of bounded variation on compact sets).

□

Exercise 1.6. (Hard, involving stopping times which will be covered later in the course, but an interesting comparison to the classical Grönwall Lemma!)**

Fix $t > 0$ and suppose that $\boldsymbol{x}, \boldsymbol{\eta}$ are real-valued, non-negative stochastic processes. Assume, moreover, that there exists constants $c', \hat{c}$ (allowed to depend on t) such that for $\mathbb{P} - a.e.\ \omega$,

$$\int_0^t \boldsymbol{\eta}_s(\omega)ds \leq c' \tag{5}$$

and for all stopping times $0 \le \theta_j < \theta_k \le t$,

$$\mathbb{E}\left(\sup_{r\in[\theta_j,\theta_k]} \boldsymbol{x}_r\right) \le \hat{c}\mathbb{E}\left(\left(\boldsymbol{x}_{\theta_j}+1\right)+\int_{\theta_j}^{\theta_k} \boldsymbol{\eta}_s \boldsymbol{x}_s ds\right) < \infty.$$

Prove that there exists a constant C dependent only on $c', \hat{c}, t$ such that

$$\mathbb{E}\left(\sup_{r\in[0,t]} \boldsymbol{x}_r\right) \le C\left(\mathbb{E}(\boldsymbol{x}_0)+1\right).$$

Proof. We shall make explicit reference to this constant $\hat{c}$, in defining a sequence of stopping times

$$\theta_j(\omega) := t \wedge \inf\left\{r \ge 0 : \int_0^r \boldsymbol{\eta}_s(\omega) ds \ge \frac{j}{2\hat{c}}\right\}$$

for $j = 0, 1, \dots$. Clearly $\theta_0 = 0$ ($\mathbb{P}-a.s.$). From the boundedness (5) uniformly over $\mathbb{P}-a.e.\ \omega$, there exists some finite N such that $\theta_N = t$ ($\mathbb{P}-a.s.$). Moreover for $0 \le j < j+1 \le N$, from the time continuity of the integral and characterisation of the first hitting times we have that

$$\int_{\theta_j}^{\theta_{j+1}} \boldsymbol{\eta}_s ds \le \frac{1}{2\hat{c}}$$

$\mathbb{P}-a.s.$ (and is in fact an equality for $j+1 < N$). From the assumed inequality we see that for any such j,

$$\begin{aligned}\mathbb{E}\left(\sup_{r\in[\theta_j,\theta_{j+1}]} \boldsymbol{x}_r\right) &\le \hat{c}\mathbb{E}\left(\left(\boldsymbol{x}_{\theta_j}+1\right)+\int_{\theta_j}^{\theta_{j+1}} \boldsymbol{\eta}_s \boldsymbol{x}_s ds\right)\\ &\le \hat{c}\mathbb{E}\left(\left(\boldsymbol{x}_{\theta_j}+1\right)+\frac{1}{2\hat{c}}\sup_{r\in[\theta_j,\theta_{j+1}]} \boldsymbol{x}_r\right)\end{aligned}$$

and therefore

$$\mathbb{E}\left(\sup_{r\in[\theta_j,\theta_{j+1}]} \boldsymbol{x}_r\right) \le 2\hat{c}\mathbb{E}\left(\boldsymbol{x}_{\theta_j}+1\right). \tag{6}$$

For $j = 0$ then

$$\mathbb{E}\left(\sup_{r\in[0,\theta_1]} \boldsymbol{x}_r\right) \le 2\hat{c}\mathbb{E}\left(\boldsymbol{x}_0+1\right)$$

and we use this along with (6) to make an inductive argument. Suppose that for some such j,

$$\mathbb{E}\left(\sup_{r\in[0,\theta_j]} \boldsymbol{x}_r\right) \le c\mathbb{E}\left(\boldsymbol{x}_0+1\right) \tag{7}$$

for a general constant c as seen throughout this proof, dependent on $c', \hat{c}, t$. Then

$$\begin{aligned}\mathbb{E}\left(\sup_{r\in[0,\theta_{j+1}]} \boldsymbol{x}_r\right) &\le \mathbb{E}\left(\sup_{r\in[0,\theta_j]} \boldsymbol{x}_r\right)+\mathbb{E}\left(\sup_{r\in[\theta_j,\theta_{j+1}]} \boldsymbol{x}_r\right)\\ &\le c\mathbb{E}\left(\boldsymbol{x}_0+1\right)+2\hat{c}\mathbb{E}\left(\boldsymbol{x}_{\theta_j}+1\right)\\ &\le c\mathbb{E}\left(\boldsymbol{x}_0+1\right)+2\hat{c}\mathbb{E}\sup_{r\in[0,\theta_j]}\left(\boldsymbol{x}_r+1\right)\\ &\le c\mathbb{E}\left(\boldsymbol{x}_0+1\right)+2\hat{c}c\mathbb{E}\left(\boldsymbol{x}_0+1\right)\\ &= c\mathbb{E}\left(\boldsymbol{x}_0+1\right)\end{aligned}$$

thanks to (6) and two applications of (7). Hence, by induction, we can conclude that (7) holds for all $j = 0, \ldots, N$ and in particular for θ_N which is $\mathbb{P} - a.s.$ equal to t. This completes the proof. $\square$